CONDUCTING LIQUID FLOW THROUGH A CHANNEL IN A NONUNIFORM MAGNETIC FIELD

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In this paper we examine the motion of an incompressible conducting liquid in a channel with non-conducting walls in the presence of a highly nonuniform magnetic field.

This problem has been studied within the framework of linear theory many times [1-6]. In the following we propose an iteration method for its solution. We study the general relations between the parameters in the zone of marked magnetic field variation and in the region of quasi-developed flow (downstream from the zone of nonhomogeneity of the magnetic field). The calculations made show that the results obtained using linear theory can also be used for finite values of the MHD interaction parameter.

1. We examine the steady motion of an inviscid, incompressible, homogeneously and isotropically conductive liquid through a channel $|x|<\infty$, $0 < y < h$ in the presence of an external magnetic field $\mathbf{B} = (0, 0, B_* b(x))$, ($B_*$=const) for small magnetic Reynolds numbers. (This "effective" field is the result of averaging the external magnetic field with respect to the z coordinate. The studied two-dimensional flow must also be considered as a three-dimensional flow which has been averaged in a definite fashion.) The system of MHD equations describing such flow has the form [1]

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+sbj_y,\quad u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}-sbj_x,\quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\qquad \left(s=\frac{\sigma B_*^2 h}{c^2\rho V}\right)\tag{1.1}$$

$$j_x=-\frac{\partial \varphi}{\partial x}+vb,\quad j_y=-\frac{\partial \varphi}{\partial y}-ub,\quad \frac{\partial j_x}{\partial x}+\frac{\partial j_y}{\partial y}=0\tag{1.2}$$

Here u, v, and i_x, j_y are the dimensionless velocity vector components and the densities of the electric field j; p and φ are the dimensionless pressure and electric potential; z, y, x are dimensionless coordinates; σ and ρ are the conductivity and density of the medium; and V = const is the velocity averaged across the channel section. We take V, $\sigma VB_*/c$, VB_*h/c and ρV^2 as the characteristic values of the velocity electric current density, potential, and pressure, respectively. We refer the coordinates to the channel height h.

The dimensionless quantity s is the MHD interaction parameter.

The system (1.1), (1.2) is supplemented by the boundary conditions at $x = -\infty$.

In the sequel we examine the following boundary conditions

$$v=0,\quad j_y=0\quad \text{for}\quad y=0,\quad y=1\tag{1.3}$$

We shall examine the case in which the external magnetic field decays rapidly outside its zone of uniformity. For a sufficiently great length of this zone the function b can be approximated by the Heaviside unit function

$$b(x)=0\quad \text{for}\quad x<0,\qquad b(x)=1\quad \text{for}\quad x>0\tag{1.4}$$

Since the flow does not interact with the magnetic field as $x \to -\infty$, the asymptotic conditions in the case in which rot V = 0 as $x \to -\infty$ have the form

$$v\to 0,\ u\to 1\quad \text{for}\quad x\to -\infty\tag{1.5}$$

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The solution of (1.1)-(1.5) was obtained in [1, 4, 6] under the assumption of smallness of s. The solution was constructed in the form of series in the parameter s.

The electric parameters were calculated in the zero approximation and the hydrodynamic parameters were calculated in the first approximation.

The aim of the present study is to construct the solution for finite s and clarify the possibility of using the linear approximation for $s \sim 1$.

For subsequent analysis it is convenient to write (1.1), (1.2) in the form

$$u\frac{\partial\omega}{\partial x}+v\frac{\partial\omega}{\partial y}=sj_x\delta(x) \tag{1.6}$$

$$\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=\omega, \qquad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1.7}$$

$$\frac{\partial j_x}{\partial y}-\frac{\partial j_y}{\partial x}=u\delta(x), \qquad \frac{\partial j_x}{\partial x}+\frac{\partial j_y}{\partial y}=0 \tag{1.8}$$

In these equations $\delta(x) = db/dx$ is the Dirac delta function and the quantity ω is the vorticity.

To obtain (1.6) we must differentiate equations (1.1) with respect to y and x and then subtract one from the other. It follows from (1.6) that vorticity is conserved along streamlines for $x < 0$ and $x > 0$. Since the vorticity equals zero for $x = -\infty$ then $\omega \equiv 0$ for $x < 0$.

The first relation (1.8) is found by differentiating (1.2).

Relations (1.6)-(1.8) show that at the section $x = 0$ several of the MHD parameters and their derivatives undergo a discontinuity. In the following this section will be considered a surface of discontinuity at which the general relations which follow from the mass and momentum conservation laws and the equations of electrodyanmics must be satisfied [4]. Thus, for $x = 0$ we have

$$[u]=0, \quad [v]=0, \quad [p]=0 \tag{1.9}$$

$$[j_x]=0, \quad [j_y]+u(0,y)=0 \tag{1.10}$$

Here the brackets [ ] denote the difference of quantities to the right and left of the section $x = 0$.

Relations (1.9) show that the parameters of an incompressible fluid traveling in a channel of constant section in the presence of volume force of finite intensity vary continuously. It can be shown that the derivatives $\partial u/\partial y$, $\partial v/\partial y$, $\partial u/\partial x$ also change continuously. However the derivatives $\partial p/\partial x$, $\partial p/\partial y$, $\partial v/\partial x$ (and consequently ω as well) undergo a discontinuity at the section $x = 0$.

Expressions (1.10) follow from continuity of the electric field component normal to the surface and the tangential component of the electric field.

Integrating (1.6) with respect to x in the limits $(-\varepsilon, +\varepsilon)$ and letting ε approach zero, with account for (1.9) we find

$$[\omega]=s\frac{j_x(0,y)}{u(0,y)} \tag{1.11}$$

Since $\omega = 0$ for $x < 0$, we obtain from (1.11) and (1.6)

$$\omega=\begin{cases}0 & (x<0)\\ [\omega] & (x=+0)\\ \Omega(x,y) & (x>0)\end{cases} \tag{1.12}$$

$$\Omega(x,y)=s\frac{j_x[0,\ y-\alpha(x,y)]}{u[0,\ y-\alpha(x,y)]}, \qquad \alpha(x,y)=\int_0^x\frac{v}{u}dx \tag{1.13}$$

The last relation in (1.12) is the integral of (1.6) for $0 < x < \infty$.

Equations (1.7) and (1.8) with account for (1.9)-(1.13) can now be written in the form

$$\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=\begin{cases}0 & (x<0)\\ \Omega(x,y) & (x>0)\end{cases}, \quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad (-\infty<x<\infty) \tag{1.14}$$

$$[u]=[v]=0 \quad (x=0), \qquad v=0 \quad (y=0,\ y=h)$$

$$\frac{\partial j_x}{\partial y}-\frac{\partial j_y}{\partial x}=0, \quad \frac{\partial j_x}{\partial x}+\frac{\partial j_y}{\partial y}=0 \quad (x\neq 0) \tag{1.15}$$

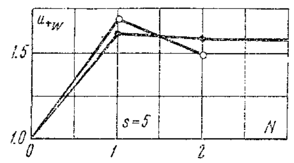

Fig. 1

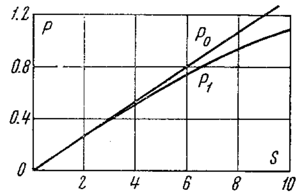

Fig. 2

The functions Ω and u in (1.14) and (1.15), respectively, are unknowns and must be found as a result of the solution of the problem.

Using (1.14), we can obtain the expression for the functions u and v in terms of Ω (by the Fourier method, for example).

Similarly, by solving the boundary value problem (1.15), in which $u(0, y)$ is considered a known function of the coordinates, we find

$$j_x = \frac{\partial F}{\partial y}, \qquad j_y = -\frac{\partial F}{\partial x}, \qquad F = -\sum_{k=1}^{\infty} \frac{u_k}{2k\pi} e^{-k\pi|x|} \sin k\pi y \tag{1.16}$$

$$u(0, y) = \sum_{k=1}^{\infty} u_k \sin k\pi y, \quad u_k = 2\int_0^1 u(0, y) \sin k\pi y \, dy$$

2. We write some relations which follow from (1.1), (1.2)

$$u_+^2(0) - 1 = 2P, \quad P = p(-\infty) - p(+\infty) \tag{2.1}$$

$$P = \int_0^1 u_+^2 dy - 1 - s \int_0^1 \int_0^y j_x(0, y) \, dy \, dy \tag{2.2}$$

$$u_+^2(1/2) - u_+^2(0) = 2s \int_0^{1/2} j_x(0, y) \, dy \tag{2.3}$$

Moreover, from (1.13)

$$\lim_{y \to 0} \frac{du_+}{dy} = s \lim_{y \to 0} \frac{j_x(0, y)}{u(0, y)} \quad (u_+ = \lim_{x \to +\infty} u(x, y)) \tag{2.4}$$

If $u(0, y)$ is known, these formulas make it possible to represent $u_+(y)$ approximately in the form of a polynomial, after isolating the singularity at the wall. Here the function $j_x(0, y)$ is known from (1.16).

The calculations made it possible to conclude that in this case the function $u_+(y)$ depends weakly on the choice of the profile $u(0, y)$ and for fixed discharge is defined primarily by the quantity x.

3. Let us examine the following iterative process. We take as the zero approximation the functions

$$\omega_0 \equiv 0, \quad u_0 \equiv 1, \quad v_0 \equiv 0, \quad j_{x0} = j_{x0}(x, y), \quad j_{y0} = j_{y0}(x, y) \tag{3.1}$$

The functions j_{x0} and j_{y0} are calculated using (1.16), in which $u(0, y) \equiv 1$. The first approximation are the functions

$$\omega_1(x, y), \ u_1, \ v_1, \ j_{x1}, \ j_{y1} \tag{3.2}$$

The functions ω_1 and Ω_1 are calculated with the aid of (1.12) and (1.13) in which $j_x = j_{x0}$, $u = u_0$, $v = v_0$. The distribution of the velocities u_1, v_1 is found from (1.14), where $\Omega = \Omega_1(x, y)$ and is calculated using the Fourier formulas or by any other method. The electric currents j_{x1} and j_{y1} are found from (1.8) and are calculated using (1.16), in which $u(0, y) = u_1(0, y)$. The second and subsequent approximations are found similarly.

We note that the quantities (3.1) coincide with terms of series having zero order of smallness in the parameter s , and the quantities (3.2) coincide with the corresponding sums of terms of zero and first order. However, when using the iterative method the parameter s is not assumed small. In principle it is possible to write out in analytic form any iteration, however, the volume of the calculations when using, for example, the Fourier method increases with the iteration number so rapidly that the use of the higher approximations becomes impossible in practice.

To estimate the rate of convergence we shall examine the following model problem. We assume that in (1.14), (1.15)

$$\Omega = s \frac{j_x(0, y)}{u(0, y)} \tag{3.3}$$

The solution of this problem corresponds to the vorticity conservation condition along the lines y = const (for x > 0) rather than along the streamlines. From the solution of (1.14) for $\Omega = \Omega(y)$ we can find that

$$u_+(y) - 1 = 2(u(0, y) - 1) \tag{3.4}$$

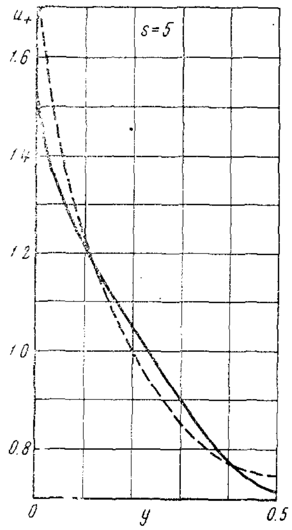

Fig. 3

Since $\Omega = du_+/dy$ as $x \to \infty$, we find from (3.3)

$$\frac{du_+}{dy} = s\frac{j_x(0, y)}{u(0, y)}, \qquad \int_0^1 u_+ \, dy = \int_0^1 u(0, y)\, dy = 1$$

Using (3.4), we obtain the equation

$$2u^\circ \frac{du^\circ}{dy} = sj_x(0, y), \quad \int_0^1 u^\circ \, dy = 1, \quad (u^\circ(y) = u(0, y)) \tag{3.5}$$

The function $j_x(0, y)$ is expressed in terms of u° with the aid of the solution (1.16).

The iterative process consists in sequential determination of the functions $j_x(0, y)$ from (1.16) and u° by solution of the nonlinear equation (3.5). We take as the initial value $u^\circ(y) \equiv 1$.

The results of the solution of the model problem are shown in Fig. 1 where the dark circles are the values of the asymptotic velocity at the wall $u_+(1) = u_+(0) = u_{+w}$, calculated on a computer with the aid of three iterations for $s = 5$. We see that the difference between the second and third approximations is so small that only two iterations are necessary for a computational accuracy of one percent.

If an error of no more than 5% is tolerable we can use only the first iteration. We note that the solution of the model problem does not cause any difficulty and the formulated algorithm permits obtaining the solution with arbitrary precision.

There is good reason to believe that the nature of the convergence of the process in solving (1.1)-(1.5) will be the same as for the model equation.

Let us find the pressure drop in the channel using (2.2). Using only the zero iteration, we find

$$P_0 = -s\int_0^1\int_0^y j_{x0}(0, y)\, dy\, dy \tag{3.6}$$

The first iteration yields

$$P_1 = \int_0^1 u_{+1}^2 \, dy - 1 - s\int_0^1\int_0^y j_{x1}(0, y)\, dy\, dy \tag{3.7}$$

The quantity P_0 corresponds to the calculation using linear theory.

The pressure losses P_0 and P_1 are shown in Fig. 2.

We see from the figure that linear theory yields quite satisfactory results up to $s = 3$-5. It is significant that the pressure drop calculated using linear theory is too high.

Let us find the asymptotic velocity profile in the second approximation (second iteration).

This can be done approximately by using (2.1)-(2.4) and not solving the problem (1.14) with Ω known from the first approximation.

After isolating the singularity at the wall of the channel, we used here the trigonometric approximation of the function.

Figure 3 shows the velocity u_{+1} (dashed) determined using linear theory [3] and the velocity u_{+2} (continuous) calculated in the second approximation. The results of the calculation of u_{+w} are also shown in Fig. 1 (open circles). We see that the nature of the iterative process convergence is the same as in the model problem.

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